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ON ASYMPTOTIC STABILITY AND INSTABILITY RELATIVE TO A PART OF VARIABLES

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We determine an analogy between certain properties of Liapunov functions and the uniform convergence of functional series and sequences satisfying the hypotheses of Dini's theorem. For autonomous systems the well-known theorems on asymptotic stability and instability relative to a part of the variables, based on the use of a Liapunov function with a sign-constant derivative, are generalized in the direction of relaxing the conditions on the set on which the derivative of the Liapunov function vanishes. We consider stability with respect to a part of the variables in the linear approximation.

1. We consider a system of differential equations of perturbed motion

$$\mathbf{x}^{\cdot} = \mathbf{X}(t, \mathbf{x}) \qquad (\mathbf{X}(t, \mathbf{0}) \equiv \mathbf{0}) \tag{1.1}$$

in which $\mathbf{x} = (y_1, \ldots, y_m, z_1, \ldots, z_p)$ is a real n-vector, $m > 0$, $p \geqslant 0$, $n = m + p$. We assume that

a) the right-hand sides of system (1.1) are continuous and satisfy the conditions for uniqueness of the solution in the region

$$t \geqslant 0, \qquad \| \mathbf{y} \| \leqslant H > 0, \qquad 0 \leqslant \| \mathbf{z} \| < \infty \tag{1.2}$$

b) the solutions of system (1.1) are z-extendable, i.e. any solution $\mathbf{x}(t)$ is defined for all $t \geqslant 0$ for which $\| \mathbf{y}(t) \| \leqslant H$.

Let $\mathbf{x} = \mathbf{x}(t; t_0, \mathbf{x}_0)$ be the solution of system (1.1) determined by the initial conditions $\mathbf{x}(t_0; t_0, \mathbf{x}_0) = \mathbf{x}_0$ (here we have adopted the notation in the survey article [1]).

Definition. The motion $\mathbf{x} = \mathbf{0}$ is said to be asymptotically $\mathbf{y}$-stable uniformly in $\mathbf{x}_0$ if it is $\mathbf{y}$-stable and if for each $t_0 \geqslant 0$ there exists $\delta(t_0) > 0$ such that

$$\| \mathbf{y}(t; t_0, \mathbf{x}_0) \| \underset{\|\mathbf{x}_0\| \leqslant \delta(t_0)}{\rightrightarrows} 0 \quad \text{as} \quad t \to \infty \tag{1.3}$$

i.e. for any $\varepsilon > 0$ we can find $T(\varepsilon, t_0) > 0$, for which $\| \mathbf{y}(t; t_0, \mathbf{x}_0) \| < \varepsilon$ follows from $\| \mathbf{x}_0 \| \leqslant \delta$ for all $t \geqslant t_0 + T$.

Note. In contrast to the definition adopted in [2], the number $\delta(t_0)$ can depend on t_0.

A general theorem on asymptotic $\mathbf{y}$-stability is proved in [3]. It turns out that the conditions of this theorem guarantee uniformity in $\mathbf{x}_0$.

Theorem 1. If a continuous function $V(t, \mathbf{x})$ is such that

$$V(t, \mathbf{x}) \geqslant a(\| \mathbf{y} \|) \qquad (a(0) = 0) \tag{1.4}$$

where $a(r)$ is a continuous function growing monotonically on $[0, H]$, and if for any $t_0 \geqslant 0$ there exists $\delta(t_0) > 0$ such that from $\|x_0\| \leqslant \delta$ follows $V(t, x(t; t_0, x_0)) \downarrow 0$ (*) as $t \to \infty$, then motion $x = 0$ is asymptotically y-stable uniformly in x_0.

Proof. Motion $x = 0$ is asymptotically y-stable [3]. Let us show that

$$V(t, x(t; t_0, x_0)) \underset{\|x_0\| \leqslant \delta(t_0)}{\rightrightarrows} 0 \quad \text{as} \quad t \to \infty \tag{1.5}$$

By hypothesis, for any $\varepsilon > 0$, $t_0 \geqslant 0$ and x_0 with $\|x_0\| \leqslant \delta(t_0)$ we can find $T(\varepsilon, t_0, x_0) > 0$, for which $V(t_0 + T, x(t_0 + T; t_0, x_0)) < \varepsilon$. By virtue of the continuity of function V and of the continuous dependence of the solution on the initial conditions, there exists a neighborhood $O(x_0)$ of point x_0 such that

$$V(t_0 + T, x(t_0 + T; t_0, x_0')) < \varepsilon \quad \text{for} \quad x_0' \in O(x_0) \tag{1.6}$$

In view of the monotonic decrease of function V, from (1.6) follows

$$V(t, x(t; t_0, x_0')) < \varepsilon \ \text{ for } t \geqslant t_0 + T(\varepsilon, t_0, x_0), \ x_0' \in O(x_0)$$

The compact region $\|x_0\| \leqslant \delta$ proves to be a covered system of neighborhoods $\{O(x_0)\}$ from which we can separate a finite subcover $O_1, \ldots, O_s$ with corresponding numbers $T_1, \ldots, T_s$. We set $T(\varepsilon, t_0) = \max\{T_1, \ldots, T_s\}$. Then $V(t, x(t; t_0, x_0)) < \varepsilon$ for all $t \geqslant t_0 + T(\varepsilon, t_0)$, provided $\|x_0\| \leqslant \delta(t_0)$, which proves (1.5). According to (1.4), (1.3) follows from (1.5). The theorem is proved.

Note. 1) Analogous reasoning can be applied [4] to systems with an infinite number of degrees of freedom.

2) Theorem 1 determines an analogy between certain properties of Liapunov functions and the uniform convergence of functional series and sequences satisfying the hypotheses of Dini's theorem [5].

Corollary 1 [6]. If system (1.1) is ω-periodic in t and its right-hand sides satisfy a Lipschitz condition in x in a neighborhood of the point $x = 0$, and if the hypotheses of Theorem 1 are fulfilled, then the motion $x = 0$ is asymptotically y-stable uniformly in $\{t_0, x_0\}$.

Indeed, by Theorem 1 there exists $\delta(0) = \delta_0 > 0$ such that

$$\|y(t; 0, x_0)\| \underset{\|x_0\| \leqslant \delta_0}{\rightrightarrows} 0 \quad \text{as} \quad t \to \infty$$

But then the motion $x = 0$ is asymptotically y-stable uniformly in $\{t_0, x_0\}$ from the region $t_0 \geqslant 0, \|x_0\| \leqslant \lambda$, where $\lambda > 0$ is such that $\|x(\omega; \tau, x_0)\| \leqslant \delta_0$, if $\tau \in [0, \omega]$, $\|x_0\| \leqslant \lambda$. Thus, in Theorem 1 of [6] we can ignore the requirement that function V be ω-periodic in t. Analogous additions are valid for the theorems in [4, 7].

Corollary 2 [4]. If function $V(t, x)$ satisfies inequality (1.4), its derivative $V^{\cdot}(t, x) \leqslant -W(t, x)$, where $W(t, x) \geqslant b(\|y\|)$ ($b(r)$ is a function of the type of $a(r)$) and $W^{\cdot} \leqslant 0$, then the motion $x = 0$ is asymptotically y-stable uniformly in x_0. If, moreover, system (1.1) is ω-periodic in t, then the asymptotic y-stability is uniform in $\{t_0, x_0\}$.

Corollary 3 [7]. If function $V(t, x)$ satisfies inequality (1.4), $V^{\cdot} \leqslant 0$, and

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*) The notation $V \downarrow 0$ means "V tends to zero, decreasing monotonically (in the wide sense)".

$V^{\cdot}(\tau, x) \leqslant -m_\eta(\tau)$ follows for any $\eta > 0$ from $V(\tau, x) \geqslant \eta$, $\| y \| \leqslant H$, moreover, if

$$\int_0^\infty m_\eta(\tau)\, d\tau = +\infty$$

then the motion $x = 0$ is asymptotically y-stable uniformly in x_0. If, besides, system (1.1) is ω-periodic in t, then the asymptotic y-stability is uniform in $\{t_0, x_0\}$.

Proof. If the hypotheses of Corollary 2 (Corollary 3) are fulfilled, then, as was shown in [4] (in [7]), for any $t_0 \geqslant 0$ there exists $\delta(t_0) > 0$ such that from $\| x_0 \| \leqslant \delta$ follows $W(t, x(t; t_0, x_0)) \downarrow 0$ $(V(t, x(t; t_0, x_0)) \downarrow 0)$ as $t \to \infty$.

Therefore, Theorem 1 and Corollary 1 are applicable; this completes the proof (*).

2. Consider the autonomous system

$$x^{\cdot} = X(x) \tag{2.1}$$

In [3, 8] criteria were proposed for system (2.1) for asymptotic y-stability and y-instability, using functions $V(x)$ with a sign-constant derivative $V^{\cdot}$ under certain requirements on the set $M = \{x\colon V^{\cdot}(x) = 0\}$. These conditions on set M can be relaxed somewhat.

Theorem 2 [3, 8]. We assume that each solution of system (2.1), starting in some neighborhood of point $x = 0$, is bounded, and let a function $V(x)$ be such that $V(x) \geqslant a(\| y \|)$, while its derivative by virtue of system (2.1)

$$V^{\cdot}(x) = 0 \quad \text{for } x \in M, \qquad V^{\cdot}(x) < 0 \quad \text{for } x \,\bar{\in}\, M \tag{2.2}$$

We denote $M_1 = \{x\colon V(x) > 0\}$, $M_0 = M_1 \cap M$. If set M_0 does not contain entire trajectories (**) for $t \in [0, \infty)$, then the motion $x = 0$ is asymptotically y-stable uniformly in $\{t_0, x_0\}$.

Proof. By virtue of the y-stability of motion $x = 0$, for any $\varepsilon \in (0, H)$ there exists $\delta(\varepsilon) > 0$ such that from $\| x_0 \| \leqslant \delta$ follows $\| y(t; 0, x_0) \| < \varepsilon$ for all $t \geqslant 0$. Let us show that from $\| x_0 \| \leqslant \delta$ ensues $V(x(t; 0, x_0)) \downarrow 0$ as $t \to \infty$. In view of $V^{\cdot} \leqslant 0$, $\lim V(x(t; 0, x_0)) = V_* \geqslant 0$ as $t \to \infty$ exists. If $V_* > 0$, then

$$V(x(t; 0, x_0)) \geqslant V_* > 0 \quad \text{for} \quad t \geqslant 0 \tag{2.3}$$

By virtue of the boundedness of the solution, $x(t_k; 0, x_0) \to x_*$ for some sequence $t_k \to \infty$; moreover, by continuity, $V(x_*) = V_*$. If we assume that $V(x(t; 0, x_*)) \equiv V_* > 0$ for $t \geqslant 0$, then $V^{\cdot}(x(t; 0, x_*)) \equiv 0$, and, consequently, $x(t; 0, x_*) \in M_1 \cap M = M_0$, which contradicts the hypothesis. Therefore, $V(x(T; 0, x_*)) < V_*$ for some $T > 0$. By virtue of the continuous dependence of the solution on the initial conditions and of the continuity of function V, for a number $T > 0$ there exists N such that for all $k > N$

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*) Added at proof-reading. Recently the author became aware of [14], published after the present paper was in press. It has turned out that in Corollary 2 the conditions on function V can be somewhat relaxed by replacing inequality (1.4) by the requirement that function V be bounded from below. In this regard the proof in [4] that $W(t, x(t; t_0, x_0)) \downarrow 0$ as $t \to \infty$, is preserved.

**) In contrast to the theorems in [3, 8], in the given case only a part of the set $M \setminus \{x = 0\}$ should not contain entire trajectories.

$$V(\mathbf{x}(T;\,0,\,\mathbf{x}(t_k;\,0,\,\mathbf{x}_0))) < V_* \tag{2.4}$$

Using the group property of autonomous systems

$$\mathbf{x}(T;\,0,\,\mathbf{x}(t_k;\,0,\,\mathbf{x}_0)) = \mathbf{x}(T + t_k;\,0,\,\mathbf{x}_0)$$

from (2.4) we obtain $V(\mathbf{x}(T + t_k;\,0,\,\mathbf{x}_0)) < V_*$, which contradicts inequality (2.3). Consequently, $V_* = 0$, whence follows the required result by Theorem 1.

Example [4, 6]. We consider the autonomous mechanical system

$$\frac{d}{dt}\frac{\partial T}{\partial q_i^{\cdot}} - \frac{\partial T}{\partial q_i} = -\frac{\partial U}{\partial q_i} + \sum_{j=1}^{n} g_{ij} q_j^{\cdot} - \frac{\partial f}{\partial q_i^{\cdot}} \qquad (i = 1, \ldots, n;\ g_{ij} = -g_{ji}) \tag{2.5}$$

Taking the total energy $H = T + U$ as the Liapunov function, we obtain

$$H^{\cdot} = -2f \tag{2.6}$$

We assume that:

1) system (2.5) has the particular solution $\mathbf{q} = \mathbf{q}^{\cdot} = 0$ (the equilibrium position);

2) the potential energy $U(q_1, \ldots, q_n)$ is positive definite relative to $q_1, \ldots, q_m$ $(m < n)$, while the dissipative function $f(q_1^{\cdot}, \ldots, q_n^{\cdot})$ is the positive definite quadratic form relative to all velocities;

3) from any mechanical considerations it is known [6] that the coordinates $q_{m+1}, \ldots, q_n$ are bounded in the perturbed motion;

4) there are no equilibrium positions in the set $U(\mathbf{q}) > 0$.

Taking (2.6) into account, on the basis of Theorem 2 we conclude that the equilibrium position $\mathbf{q} = \mathbf{q}^{\cdot} = 0$ is asymptotically stable relative to $q_1, \ldots, q_m, q_1^{\cdot}, \ldots, q_n^{\cdot}$ uniformly in $\{t_0, \mathbf{q}_0, \mathbf{q}_0^{\cdot}\}$. In this example the corresponding theorems in [3, 8] are not applicable: the set $H^{\cdot} = 0$ can contain entire trajectories other than $\mathbf{q} = \mathbf{q}^{\cdot} = 0$, since the equilibrium position $\mathbf{q} = \mathbf{q}^{\cdot} = 0$ is, in general, not isolated.

Theorem 2 ceases to be true if we ignore the requirement of boundedness of the solutions, which is shown by example of the system

$$y^{\cdot} = -\frac{y}{1 + z^2}, \quad z^{\cdot} = z$$

The general solution of this, unbounded in z, has the form

$$y = y_0 \exp\left[-\int_0^t \frac{d\tau}{1 + z_0^2 \exp(2\tau)}\right], \qquad z = z_0 \exp(t)$$

The solution $y = z = 0$ is not asymptotically y-stable since for $z_0 \neq 0$,

$$\int_0^\infty \frac{d\tau}{1 + z_0^2 \exp(2\tau)} < +\infty$$

However, the function $2V = y^2$ satisfies the hypotheses of Theorem 2. Indeed, the set $M = \{(y, z) : V^{\cdot} = 0\}$ is the axis $y = 0$. For $y \neq 0$ we have $V > 0$. Therefore, the intersection $M \cap \{(y, z) : V > 0\}$ is empty and, consequently, does not contain entire trajectories.

Theorem 3 [8]. We assume that: (1) each solution of system (2.1), starting in some neighborhood of point $\mathbf{x} = 0$, is $\mathbf{z}$-bounded; (2) the function $V(\mathbf{x})$ is such that $V(0) = 0$ and in any neighborhood of the origin there exist points $\mathbf{x}$ for which

$V(\mathbf{x}) < 0$; (3) The derivative $V^{\cdot}$ satisfies condition (2.2). We denote $M_1 = \{\mathbf{x}: V(\mathbf{x}) < 0\}$, and $M_0 = M_1 \cap M$. If M_0 does not contain entire trajectories for $t \in [0, \infty)$, then motion $\mathbf{x} = \mathbf{0}$ is y-unstable.

Proof. We assume the contrary and we select $\mathbf{x}_0$ from the conditions $V(\mathbf{x}_0) < 0$, $\| \mathbf{y}(t; 0, \mathbf{x}_0) \| < H$ for $t \geqslant 0$. Then

$$V(\mathbf{x}(t; 0, \mathbf{x}_0)) \leqslant V(\mathbf{x}_0) < 0 \tag{2.7}$$

and, consequently, $\| \mathbf{x}(t; 0, \mathbf{x}_0) \| \geqslant \eta > 0$. The set Γ^+ of ω-limit points of the solution $\mathbf{x}(t; 0, \mathbf{x}_0)$ is not empty (by virtue of the boundedness of the solutions) and is invariant [9], where $\Gamma^+ \subset M$ [10, 11]; by virtue of (2.7), $\Gamma^+ \subset M_1$. Thus, $\Gamma^+ \subset M_0 = M_1 \cap M$. Consequently, set M_0 contains a trajectory, which is impossible. The theorem is proved.

Theorem 4 [8]. Let conditions (1) – (3) of Theorem 3 be fulfilled, as well as (4) $V(0, \mathbf{z}) \geqslant 0$ for any $\mathbf{z}$; (5) the set $\{\mathbf{x}: \mathbf{y} = \mathbf{0}\}$ is invariant. We denote $M_1 = \{\mathbf{x}: V(\mathbf{x}) < 0\}$, $M_0 = M_1 \cap (M \setminus \{\mathbf{x}: \mathbf{y} = \mathbf{0}\})$. If M_0 does not contain entire trajectories for $t \in [0, \infty)$, then motion $\mathbf{x} = \mathbf{0}$ is y-unstable.

Proof. We assume the contrary and we select $\mathbf{x}_0$ just as in the proof of Theorem 3. The set Γ^+ is not empty. Let $\lim_{n\to\infty} \mathbf{x}(t_n; 0, \mathbf{x}_0) = \mathbf{x}_* \in \Gamma^+$

If $\lim \| \mathbf{y}(t; 0, \mathbf{x}_0) \| = 0$ as $t \to \infty$, then $\mathbf{y}_* = \mathbf{0}$ and, by passing to the limit in the inequalities

$$\lim_{t\to\infty} V(\mathbf{x}(t; 0, \mathbf{x}_0)) \leqslant V(\mathbf{x}_0) < 0$$

we obtain $0 \leqslant V(\mathbf{0}, \mathbf{z}_*) < V(\mathbf{x}_0)$, which is impossible. Consequently, $\| \mathbf{y}(t_n; 0, \mathbf{x}_0) \| \geqslant \eta > 0$ for some sequence $t_n \to \infty$, and we can assume that $\mathbf{y}_* \neq \mathbf{0}$. According to (5), $\| \mathbf{y}(t; 0, \mathbf{x}_*) \| \neq 0$ for all $t \geqslant 0$, whence, by virtue of the invariance of Γ^+ and of the properties $\Gamma^+ \subset M$ and $\Gamma^+ \subset M_1$, follows $\mathbf{x}(t; 0, \mathbf{x}_*) \in M_1 \cap (M \setminus \{\mathbf{x}: \mathbf{y} = \mathbf{0}\})$ for any $t \geqslant 0$, which is impossible. The theorem is proved.

3. We consider the linear system

$$\mathbf{x}^{\cdot} = L\mathbf{x} \tag{3.1}$$

where L is a constant matrix. The following theorem is known.

Theorem A (*). For the solution $\mathbf{x} = \mathbf{0}$ of system (3.1) be asymptotically stable in the m variables $y_1, \ldots, y_m$ (**), it is necessary and sufficient that system (3.1) have the form

$$\mathbf{y}^{\cdot} = A\mathbf{y}, \qquad \mathbf{z}^{\cdot} = B\mathbf{y} + C\mathbf{z} \tag{3.2}$$

(A, B and C are matrices of appropriate orders), and that the roots of the equation $\det(A - \lambda E) = 0$ have negative real parts.

Consider the perturbed system

$$\mathbf{y}^{\cdot} = A\mathbf{y} + \mathbf{f}(t, \mathbf{y}, \mathbf{z}), \qquad \mathbf{z}^{\cdot} = B\mathbf{y} + C\mathbf{z} + \mathbf{g}(t, \mathbf{y}, \mathbf{z}) \tag{3.3}$$

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*) Pfeiffer, K., La méthode directe de Liapounoff [Liapunov] appliquée à l'étude de la stabilité partielle. Dissertation, Univ. Catholique de Louvain, Faculté des Sciences, 1968.

**) In the sense of the theorem it is assumed here that the solution $\mathbf{x} = \mathbf{0}$ is not assymptotically stable in more than m variables.

Theorem 5. If $\operatorname{Re}\lambda_j(A) < 0$ and if in region (1.2)

$$\|\mathbf{f}(t, \mathbf{y}, \mathbf{z})\| \leqslant h \|\mathbf{y}\| \tag{3.4}$$

where h is a sufficiently small constant, then the motion $\mathbf{x} = \mathbf{0}$ of system (3.3) is exponentially asymptotically $\mathbf{y}$-stable uniformly in $\{t_0, \mathbf{y}_0\}$ in-the-large with respect to $\mathbf{z}_0$, i.e.

$$\|\mathbf{y}(t; t_0, \mathbf{x}_0) \leqslant M \|\mathbf{y}_0\| \exp [-\alpha (t - t_0)]$$
$$(M > 0, \alpha > 0 - \text{const}, 0 \leqslant \|\mathbf{z}_0\| < \infty)$$

Proof. By hypothesis, $\operatorname{Re}\lambda_j(A) < 0$, therefore, according to Liapunov's theorem the equation $\operatorname{grad} V(\mathbf{y}) \cdot A\mathbf{y} = -\|\mathbf{y}\|^2$ has a single-valued solution as a positive-definite quadratic form $V(\mathbf{y})$. Its derivative, by virtue of (3.3), is

$$V^\cdot(t, \mathbf{y}, \mathbf{z}) = -\|\mathbf{y}\|^2 + \operatorname{grad} V(\mathbf{y}) \cdot \mathbf{f}(t, \mathbf{y}, \mathbf{z})$$

For a sufficiently small h we have [12] $V^\cdot \leqslant -\beta V$ ($\beta = \text{const} > 0$), whence follows the result required.

Note. According to (3.4), $\mathbf{f}(t, \mathbf{0}, \mathbf{z}) \equiv \mathbf{0}$, therefore, the condition [13] $\mathbf{Y}(t, \mathbf{0}, \mathbf{z}) \equiv \mathbf{0}$ is fulfilled here.

Condition (3.4) is easily verified if the space $\mathbf{R_z}$ is compact. However, it is very hard if an unbounded region (1.2) is considered. This inconvenience is removed if the $\mathbf{z}$-boundedness of the solutions is known in advance. We recall [7] that the solutions of system (3.3) are said to be $\mathbf{z}$-bounded uniformly in $\{t_0, \mathbf{x}_0\}$ if for any compactum $K \subset \mathbf{R_x}$ there exists a constant $N(K)$ such that

$$\|\mathbf{z}(t; t_0, \mathbf{x}_0)\| \leqslant N \quad \text{as} \quad t \geqslant t_0 \tag{3.5}$$

follows from $t_0 \geqslant 0$, $\mathbf{x}_0 \in K$. A criterion for such boundedness is given in [7].

Theorem 6. If condition (3.5) is fulfilled, where $K = \{\mathbf{x}: \|\mathbf{x}\| \leqslant \delta\}$ with a sufficiently small $\delta > 0$, $\operatorname{Re}\lambda_j(A) < 0$, and

$$\|\mathbf{f}(t, \mathbf{y}, \mathbf{z}) \leqslant h \|\mathbf{y}\| \quad \text{as} \quad t \geqslant 0, \|\mathbf{z}\| \leqslant N \tag{3.6}$$

($h = \text{const} > 0$ is sufficiently small), then the motion $\mathbf{x} = \mathbf{0}$ of system (3.3) is exponentially asymptotically $\mathbf{y}$-stable.

Proof. Having chosen the function $V(\mathbf{y})$ just as in the proof of Theorem 5, for the solutions $\mathbf{x}(t; t_0, \mathbf{x}_0)$ with $\|\mathbf{x}_0\| \leqslant \delta$ we obtain ($\beta = \text{const} > 0$)

$$\frac{d}{dt} V(\mathbf{y}(t; t_0, \mathbf{x}_0)) = V^\cdot(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) \leqslant -\beta \|\mathbf{y}(t; t_0, \mathbf{x}_0)\|^2$$

for a sufficiently small h, whence follows the required result.

Condition (3.6) is fulfilled for a wide class of functions, for example, for the polynomials (the sum is finite)

$$f_j(t, \mathbf{y}, \mathbf{z}) = \sum a^{(j)}_{i_1 \ldots i_m k_1 \ldots k_p} y_1^{i_1} \ldots y_m^{i_m} z_1^{k_1} \ldots z_p^{k_p}$$

with $i_1 + \ldots + i_m \geqslant 2$, continuous and bounded coefficients a.

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REFERENCES

1. Oziraner, A. S. and Rumiantsev, V. V., The method of Liapunov functions in the stability problem for motion with respect to a part of the variables. PMM Vol. 36, №2, 1972.
2. Krasovskii, N. N., Stability of Motion. Stanford Univ. Press, Stanford, Cal., (Translation from Russian), 1963.
3. Rumiantsev, V. V., On asymptotic stability and instability of motion with respect to a part of the variables. PMM Vol. 35, №1, 1971.
4. Oziraner, A. S., On the stability of equilibrium positions of a solid body with a cavity containing a liquid. PMM Vol. 36, №5, 1972.
5. Fikhtengol'ts, G. M., Course on Differential and Integral Calculus, Vol. 2. Moscow, "Nauka", 1966. (See also: Fundamentals of Mathematical Analysis Vol. 2, Pergamon Press, Book №10060, 1965).
6. Oziraner, A. S., On asymptotic stability relative to a part of the variables. Vestn. Mosk. Gos. Univ., Ser. Matem. Mekhan., №1, 1972.
7. Oziraner, A. S., On certain theorems of Liapunov's second method. PMM Vol. 36, №3, 1972.
8. Risito, C., Sulla stabilità asintotica parziale. Ann. di Math. purra ed appl., Ser. 4, Vol. 84, 1970.
9. Nemytskii, V. V. and Stepanov, V. V., Qualitative Theory of Differential Equations. Princeton, N. J., Princeton Univ. Press, 1960.
10. La Salle, J. and Lefschetz, S., Stability by Liapunov's Direct Method. New York, Academic Press, 1961.
11. Barbashin, E. A., Introduction to Stability Theory. Moscow, "Nauka", 1967.
12. Malkin, I. G., Theory of Stability of Motion. Moscow, "Nauka", 1966.
13. Oziraner, A. S., On the question of motion stability relative to a part of the variables. Vestn. Mosk. Gos. Univ., Ser. Matem. Mekhan., №1, 1971.
14. Salvadori, L., Sul problema della stabilità asintotica. Meccanica. J. of Italian Association of Theor. and Appl. Mec., Vol. 7, №4, 1972.

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ON THE OPTIMAL SPACING OF MEASUREMENTS IN THE METHOD OF LEAST SQUARES

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We consider the problem of the distribution of a specified number of measurments on a given interval, ensuring the least variance of the estimate of one of the parameters linearly related with the function being measured. Assuming a normal distribution law for the measurement errors, we derive equations describing necessary extremum conditions for the corresponding variance. Using